Ex: Compute $\int_C (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy$ for C the picture.

Circle $x^2+y^2=9$

solve: $\int_{\partial D} (3y - e^{\sin(x)})dx + (7x+\sqrt{y^4+1})\,dy$

$$= \iint_D \frac{\partial}{\partial x}\left(7x+\sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin(x)}\right) dA$$

$$= \iint_D 7 - 3 \, dA = 4 \iint_D dA \qquad 4\cdot(\pi(3)^2) = 36\pi$$

11/12/2021

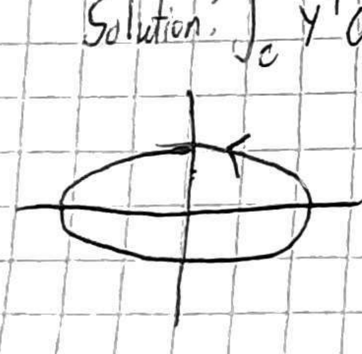
Last time: Green's Theorum

Proposition (Green's Theorum): Suppose D a connected region in the plane with ∂D a smooth simple closed curve. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on some open region R containing D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Ex: Compute $\int_C y^4\,dx + 2xy^2\,dy$ for C the ellipse $x^2+2y^2=2$

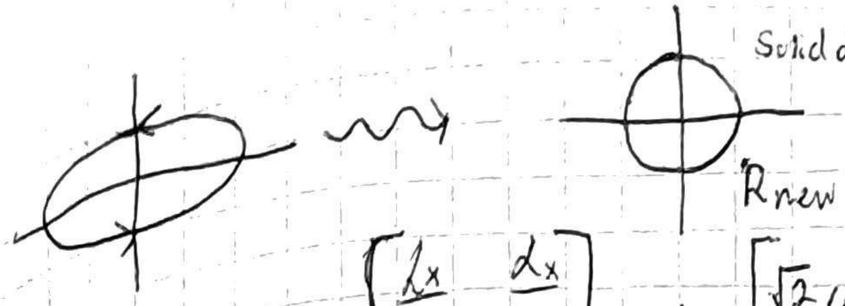
Solution: $\int_C y^4\,dx + 2xy^2\,dy = \iint_D \left(\frac{\partial}{\partial x}[2xy] - \frac{\partial}{\partial y}[y^4]\right) dA$ D = "solid ellipse"



$$\begin{cases} x = \sqrt{2}\,r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \text{ for } \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$x^2+2y^2 = 2 \Rightarrow 2r^2\cos^2(\theta) + 2r^2\sin^2(\theta) = 2 \Rightarrow$$

$$r^2 = 1 \Rightarrow r = 1$$



$$\frac{\partial(x,y)}{\partial(r,\theta)} = \text{determinant} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \sqrt{2}\cos(\theta) & -\sqrt{2}\, r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \Rightarrow$$

$$= \sqrt{2}\, r\cos^2(\theta) - \sqrt{2}\, r\sin^2(\theta) = \sqrt{2}\, r$$

$$\therefore \iint_D (2y^2 - 4y^3)\, dA = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(2(r\sin(\theta))^2 - 4(r\sin(\theta))^3\right) \cdot \sqrt{2}\, r\, d\theta\, dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 2\sqrt{2}\, r^3 \left(\sin^2(\theta) - 2r\sin^3(\theta)\right) d\theta\, dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 2\sqrt{2}\, r^3 \sin^2(\theta)\left(1 - 2r\sin(\theta)\right) d\theta\, dr$$

$$= \int_{r=0}^{1} 2\sqrt{2}\, r^3 \underbrace{\int_{\theta=0}^{2\pi} (1-\cos^2(\theta))(1 - 2r\sin(\theta))\, d\theta}\, dr$$

$u = \cos(\theta)$
$du = -\sin(\theta)\, d\theta$

$$\hookrightarrow \int_{\theta=0}^{2\pi} (1-\cos^2(\theta))(1-2r\sin(\theta))\, d\theta \Rightarrow \int_{\theta=0}^{2\pi} 1-\cos^2(\theta)\, d\theta - \int_{\theta=0}^{2\pi} 2r(1-\cos^2(\theta))\sin(\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(1 - \left(\tfrac{1}{2}(1+\cos(2\theta))\right)\right) d\theta - 2r\int_{\theta=0}^{2\pi} -(1-u^2)\, du \Rightarrow \quad \cos^2(\theta) = \tfrac{1}{2}(1+\cos(2\theta))$$

$$= \int_{\theta=0}^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\right) d\theta + 2r\left[u - \tfrac{1}{3}u^3\right]_{\theta=0}^{2\pi} \Rightarrow$$

$$= \left[\tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\right]_{\theta=0}^{2\pi} + 2r\left[\cos(\theta) - \tfrac{1}{3}\cos^3(\theta)\right]_{\theta=0}^{2\pi} \Rightarrow$$

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$$=\left(\frac{1}{2}(2\pi-0)-\frac{1}{4}(\sin(4\pi)-\sin(0)\right)+2r\left((\cos(2\pi)-\cos(0))-\frac{1}{3}(\cos^3(2\pi)-\cos^3(0)\right)$$

$$=\pi-\frac{1}{4}(0)-2r\left(0-\frac{1}{3}\cdot 0\right)=\pi$$

Outer Integral

$$\int_{r=0}^{1} 2\sqrt{2}\,r^3\pi\,dr=\frac{2\sqrt{2}\pi}{4}\left[r^4\right]_{r=0}^{1}=\frac{\pi}{\sqrt{2}}(1-0)=\frac{\pi}{\sqrt{2}}$$

$$\therefore \int_C y^4\,dx+2xy^2\,dy=\frac{\pi}{\sqrt{2}}$$

Note: The way we've been using Green's theorem has been turn a line integral into a double integral. But we can go the other way!

If our double Integral's region is "nice enough," then we can turn our double Integral into a line integral.

Note: If P and Q satisfy $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1$ then

$$\text{Area}(D)=\iint_D 1\,dA=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\int_{\partial D} P\,dx+Q\,dy$$

Ex. Compute the area of the general ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution: $\text{Area}(D)=\int_D P\,dx+Q\,dy$ when P, Q satisfy $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1$

Choose $Q(x,y)=x$ and $P(x,y)=0$ then $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial}{\partial x}[x]-\frac{\partial}{\partial y}[0]=1$

$$\therefore \text{Area}(D)=\int_{\partial D} 0\,dx+x\,dy=\int_{\partial D} x\,dy$$

∂D is parameterized by $\bar{r}(\theta) = \langle a\cos(\theta), b\sin(\theta) \rangle$ on $0 \le \theta \le 2\pi$

$\therefore$ Area$(D) = \int_{\partial D} x\, dy = \int_{\theta=0}^{2\pi} a\cos(\theta)\, b\cos(\theta)\, d\theta = ab\int_{\theta=0}^{2\pi} \cos^2(\theta)\, d\theta \Rightarrow$

$ab\int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos(2\theta))\,d\theta \Rightarrow \cancel{ab\frac{1}{2}\int_{\theta=0}^{2\pi}}\ ab\frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} \Rightarrow$

$\frac{1}{2}ab\left((2\pi - 0) + \frac{1}{2}(0-0)\right) = ab\pi$

16.5: Curl and Divergence

Goal: Define and study two new operations on vector fields.

Definition: The curl of a vector field $\bar{v}$ on $\mathbb{R}^3$ is the vector field

"$\nabla \times \bar{v}$" $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \bar{v} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle P, Q, R \rangle$

$= \text{determinant}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$

Ex: Compute curl$(\bar{v})$ for $\bar{v}\ \langle xy, xyz, -y^2 \rangle$

curl$(\bar{v}) = \nabla \times \bar{v} = \text{determinant}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xyz & -y^2 \end{bmatrix} = \left\langle \begin{matrix} \frac{\partial}{\partial y}[-y^2] - \frac{\partial}{\partial z}[xyz], \\ -\left(\frac{\partial}{\partial x}[-y^2] - \frac{\partial}{\partial z}[xy]\right), \\ \frac{\partial}{\partial x}[xyz] - \frac{\partial}{\partial y}[xy] \end{matrix} \right\rangle$

$= \langle -2y - xy, 0, yz - x \rangle$

Observation. If $\bar{v} = \nabla f$ is a conservative vector field, then $\bar{v} = \langle f_x, f_y, f_z \rangle$

and Curl$(\bar{v}) = \nabla \times \bar{v} = \text{determinant}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{bmatrix}$ (Converse ~~not~~ also true)

$= \langle f_{zy} - f_{yz}, -(f_{xz} - f_{zx}), f_{yx} - f_{xy} \rangle = \vec{0}$ by Clairaut's Theorem!

Point: A vector field $\bar{v}$ is conservative if and only if $\text{Curl}(\bar{v}) = \vec{0}$

Definition: The divergence of vector field $\bar{v} = \langle v_1, v_2, \ldots, v_n \rangle$

$$\text{div}(\bar{v}) = "\nabla \cdot \bar{v}" = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \langle v_1, v_2, \ldots, v_n \rangle$$

$$= \left(\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \ldots \frac{\partial v_n}{\partial x_n} \right)$$

Ex: $\text{div}(\langle xy, xyz, -y^2 \rangle) =$

$$= \nabla \cdot \langle xy, xyz, -y^2 \rangle = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[-y^2]$$

$$= y + xz + 0 = y + xz$$

Proposition: A vector field $\bar{v}$ is the curl of some vector field $\bar{w}$ if and only if $\text{div}(\bar{v}) = 0$

$(\Rightarrow)$: If $\bar{v} = \text{curl}(\langle P, Q, R \rangle)$ then we have

$$\bar{v} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

then $\text{div}(\bar{v}) = \nabla \cdot \bar{v} = \frac{\partial}{\partial x}[R_y - Q_z] + \frac{\partial}{\partial y}[P_z - R_x] + \frac{\partial}{\partial z}[Q_x - P_y]$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{yz} - P_{xz}$$

$$= (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) = 0 + 0 + 0 = 0$$

by Clairaut's theorem

Point: Proposition can be used to check if a vector field is a curl.